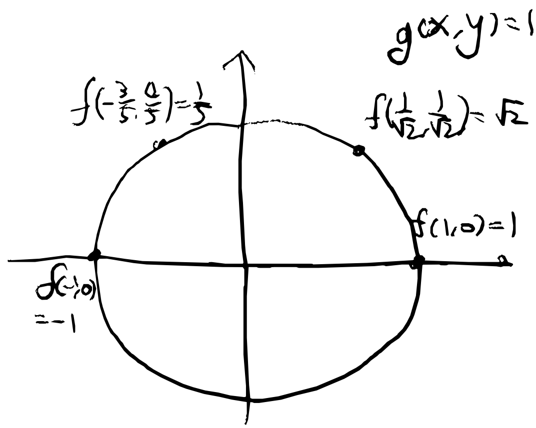
Lagrange multipliers

Typical problem Want to find the max/min of $f(x,y)$ on the curve $g(x,y)=k$.

Example What is the max/min values of

$$f(x,y)=x+y$$

on the circle $g(x,y)=x^2+y^2=1$?



<u>Lagrange multipliers</u> The max/min of $f(x,y)$ on the curve $g(x,y)=k$ is achieved when the two vectors

$\nabla f(x,y)$ and $\nabla g(x,y)$ are parallel.

Such points are called <u>Lagrange critical points</u>.

This happens in two cases:

① If $\nabla g(x,y)=\vec{0}$ (any vector is parallel to $\vec{0}$)

② If there is a scalar λ such that

$$\nabla f(x,y)=\lambda\nabla g(x,y).$$

This λ is called <u>Lagrange multiplier</u>.

<u>Back to the example</u> From $f(x,y)=x+y$ and $g(x,y)=x^2+y^2$,

$$\nabla f(x,y)=\langle 1,1\rangle$$

$$\nabla g(x,y)=\langle 2x,2y\rangle.$$

$\langle 1,1\rangle$ and $\langle 2x,2y\rangle$ are parallel means that $2x=2y$, or $x=y$. On the circle $x^2+y^2=1$, $x=y$ implies that $x^2+x^2=1$, or $x^2=\frac{1}{2}$, or $x=\pm\sqrt{\frac{1}{2}}$.

$$x=\sqrt{\tfrac{1}{2}} \Rightarrow y=\sqrt{\tfrac{1}{2}} \Rightarrow f\left(\sqrt{\tfrac{1}{2}},\sqrt{\tfrac{1}{2}}\right)=2\sqrt{\tfrac{1}{2}}=\sqrt{2} \leftarrow \text{max}$$

$$x=-\sqrt{\tfrac{1}{2}} \Rightarrow y=-\sqrt{\tfrac{1}{2}} \Rightarrow f\left(-\sqrt{\tfrac{1}{2}},-\sqrt{\tfrac{1}{2}}\right)=-2\sqrt{\tfrac{1}{2}}=-\sqrt{2} \leftarrow \text{min}$$

The method of Lagrange multipliers is the final piece we need to go through the four-step process, especially at step 3, when dealing with the boundaries.

$\boxed{\text{Example}}$ Find the global max/min values of

$f(x,y) = x+y$ on the domain $D := \{x^2+y^2 \le 1\}$.

$\boxed{\text{Step 1}}$ Find the critical points.

$f(x,y) = x+y$, so $\nabla f(x,y) = \langle 1,1 \rangle$, which is never $\vec{0}$.

$\Rightarrow$ no critical points

$\boxed{\text{Step 2}}$ Find the max/min values of $f(x,y)$ at the critical points : <u>vacuous</u>. (no critical points)

$\boxed{\text{Step 3}}$ Find the max/min values of $f(x,y)$ on the boundary.

What do we need to do here? We need to find the max/min values of $f(x,y) = x+y$ on the boundary, so $x^2+y^2=1$. This is exactly what we did in the previous example!

<u>Max : $\sqrt{2}$, Min : $-\sqrt{2}$</u>

$\boxed{\text{Step 4}}$ Compare.

	Critical	Boundary
Max	N/A	$\sqrt{2}$
Min	N/A	$-\sqrt{2}$

Global max: $\sqrt{2}$

Global min: $-\sqrt{2}$.

$\boxed{\text{Example}}$ Find the global max/min of $f(x,y) = x^2 + 2y^2$ on the domain $D = \{x^2+y^2 \le 1\}$.

$\boxed{\text{Step 1}}$ Find the critical points.

$\nabla f(x,y) = \langle 2x, 4y \rangle$, so $\nabla f(x,y) = \langle 0,0 \rangle$ happens when $x=y=0$.

$\Rightarrow (0,0)$ is the only critical point.

$\boxed{\text{Step 2}}$ Find the max/min values of $f(x,y)$ at the critical points.

$f(0,0)=0$ is both the max/min value at the critical points, because $(0,0)$ is the only critical point.

$\boxed{\text{Step 3}}$ Find the max/min values of $f(x,y)$ on the boundary.

The boundary of $D = \{x^2+y^2 \le 1\}$ is the circle $x^2+y^2=1$.
We want to use Lagrange multipliers, so let's express the equation for the boundary as $g(x,y)=1$, where $g(x,y) = x^2+y^2$.

We therefore need to solve:

$\boxed{\text{Find the max/min of } f(x,y) = x^2+2y^2 \text{ on } g(x,y)=1,\ g(x,y)=x^2+y^2.}$

By the method of Lagrange multipliers,

Lagrange critical points are when $\boxed{\nabla g(x,y) = \langle 0,0 \rangle}$ OR

$\nabla f(x,y) = \lambda \nabla g(x,y)$ for some λ.

$\nabla g(x,y) = \langle 2x, 2y \rangle$.

* $\boxed{\nabla g(x,y) = \langle 0,0 \rangle \text{ case}}$ $\rightsquigarrow$ $x=0,\ y=0$. This point is not on the domain $x^2+y^2=1$, so out of consideration.

* $\nabla f(x,y) = \lambda \nabla g(x,y)$ case $\rightsquigarrow$ $\langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle$

$\rightsquigarrow$ $2x = 2\lambda x$ and $4y = 2\lambda y$.

We are thus solving system of equations

① $2x = 2\lambda x$
② $4y = 4\lambda y$
③ $x^2+y^2=1$

in variables λ, x, y.

① $2x = 2\lambda x$ means that $2x - 2\lambda x = 0$ means that

$2x(1-\lambda) = 0$ means that

either $x=0$ OR $1-\lambda=0$.

If $x=0$, then ③ $x^2+y^2=1$ implies that $y^2=1$, $y=\pm1$.
Indeed, then ② $4y=2\lambda y$ will be satisfied with $\lambda=\pm2$.

$\rightsquigarrow$ Lagrange critical points:

$(0,1),\ (0,-1)$

If $1-\lambda=0$, then $\lambda=1$, so ② $4y=2\lambda y$ implies that $4y=2y$, so $y=0$. By ③ $x^2+y^2=1$, this implies that $x^2=1$, $x=\pm1$.

$\rightsquigarrow$ Lagrange critical points

$(1,0),\ (-1,0)$.

We found that $(0,1), (0,-1), (1,0), (-1,0)$ are the Lagrange critical points.

$f(0,1)=2$, $f(0,-1)=2$, $f(1,0)=1$, $f(-1,0)=1$,

so on the boundary, the max value of $f(x,y)$ is 2,
the min value of $f(x,y)$ is 1.

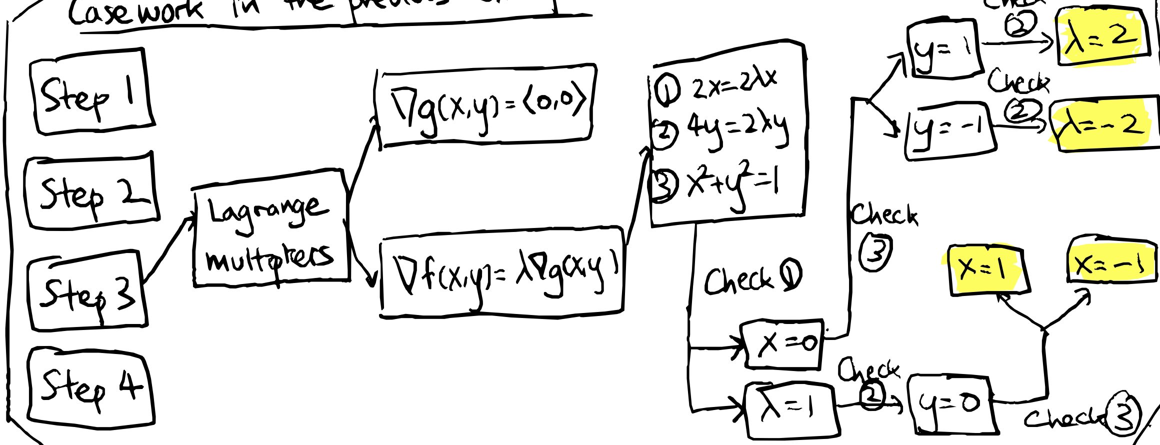
$\boxed{\text{Step 4}}$ Compare.

	Critical	Boundary
Max	0	2
Min	0	1

The global max is 2, the global min is 0.

To use the Lagrange multipliers, there are a lot of bookkeeping. When you are solving a system of equations, make sure that you check all cases and all equations.

Casework in the previous example



In the above diagram, notice that we always check all the equations ①, ②, ③ once before reaching the highlighted ends.

The method of Lagrange multipliers we learned works even if there is an equality in the definition of the domain. The 4-step-process is the same, except that we use Lagrange critical points instead of just critical points.

4-step-process when there is an equality in the domain

$\boxed{\text{Objective}}$: Find the global max/min of $f(x,y)$ on the domain defined by $\left\{ \begin{array}{l} \cdots \text{ bunch of inequalities } \cdots \\ \& \quad g(x,y)=k \end{array} \right\}$.

$\boxed{\text{Step 1}}$ Find the Lagrange critical points.
(Namely, find (x,y) such that either $\nabla g(x,y) = \langle 0,0 \rangle$ or there is λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$.)

$\boxed{\text{Step 2}}$ Find the max/min values of $f(x,y)$ at the Lagrange critical points.

$\boxed{\text{Step 3}}$ Find the max/min values of $f(x,y)$ on the boundary.

$\boxed{\text{Step 4}}$ Compare the values from Steps 2 & 3. Take the largest/smallest.

The above 4-step process works in the presence of an equality condition for the domain, regardless of the number of variables.

Let's see how the 4-step process can be used for 3-variable case.

Example. Find the global max/min of $f(x,y,z) = 2x+2y+z$ on the domain $D = \{(x,y,z)$ such that $x^2+y^2+z^2 \le 9\}$.

Firstly, in the setup of the Example, the domain is defined by an <u>inequality</u> $x^2+y^2+z^2 \le 9$. So, in the absence of an equality in the definition of the domain, the 4-step process should look for <u>critical points</u> (as opposed to <u>Lagrange critical points</u>).

$\boxed{\text{Step 1}}$ Find the <u>critical points</u>.

$\nabla f(x,y,z) = \langle 2,2,1 \rangle$ is never zero, so there is no critical point.

$\boxed{\text{Step 2}}$ $\Rightarrow$ Vacuous (no critical points)

$\boxed{\text{Step 3}}$ Find the max/min values of $f(x,y,z)$ on the boundary.

What is the boundary of the domain $D = \{x^2+y^2+z^2 \le 9\}$?

It is the shell of the sphere, $\{x^2+y^2+z^2 = 9\}$.

Thus Step 3 gives rise to a <u>new optimization sub-problem</u>:

Find the global max/min of $f(x,y,z) = 2x+2y+z$ on the domain $\{x^2+y^2+z^2 = 9\}$.

Let $g(x,y,z) = x^2+y^2+z^2$, so that the new domain is $g(x,y,z) = 9$. Now the new domain is defined using an <u>equality</u>, so the upcoming 4-step process should look for the <u>Lagrange critical points</u>.

$\boxed{\text{Step 3-1}}$ Find the Lagrange critical points.

This happens either when $\nabla g(x,y,z) = \langle 0,0,0\rangle$ OR when $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for some λ.

- If $\nabla g(x,y,z) = \langle 0,0,0\rangle$, then $\nabla g(x,y,z) = \langle 2x, 2y, 2z\rangle$, So $x=y=z=0$. This point, $(x,y,z)=(0,0,0)$, is not on the domain $x^2+y^2+z^2=9$, so this is out of consideration.
- If $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$, then $\langle 2,2,1\rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle \rightsquigarrow$ we get 3+1 equations

① $2=2\lambda x$

② $2=2\lambda y$

③ $1=2\lambda z$

④ $x^2+y^2+z^2=9$

Using ① $2=2\lambda x$, we know that $\cdot\ \lambda, x$ are not zero, and $\cdot\ \lambda = \frac{1}{x}$. We can plug this into ② $2=2\lambda y$ to get $2=\frac{2y}{x}$, or $y=x$. We can also plug into ③ $1=2\lambda z$ to get $1=\frac{2z}{x}$, or $z=\frac{x}{2}$. We can plug these points into ④ $x^2+y^2+z^2=9$ to get

$x^2+x^2+\left(\frac{x}{2}\right)^2=9$, or $\frac{9}{4}x^2=9$, or $x^2=4$, so

$x=2 \Rightarrow y=x=2,\ z=\frac{x}{2}=1 \Rightarrow (2,2,1)$

OR

$x=-2 \Rightarrow y=x=-2,\ z=\frac{x}{2}=-1 \Rightarrow (-2,-2,-1)$

Lagrange crit. pt.s

$\boxed{\text{Step 3-2}}$ Find the max/min values of $f(x,y,z)$ at the Lagrange critical points.

$f(2,2,1) = 2\cdot 2 + 2\cdot 2 + 1\cdot 1 = 9 \leftarrow$ Max

$f(-2,-2,-1) = 2\cdot(-2) + 2\cdot(-2) + 1\cdot(-1) = -9 \leftarrow$ Min

Step 3-3 Find the max/min values of $f(x,y,z)$ on the boundary.

Recall that the domain is $\{x^2+y^2+z^2=9\}$, the shell of a sphere. This has no boundary. $\Rightarrow$ <u>Vacuous</u>

Step 3-4 Compare.

	Lagrange critical points	Boundary
Max	9	X
Min	-9	X

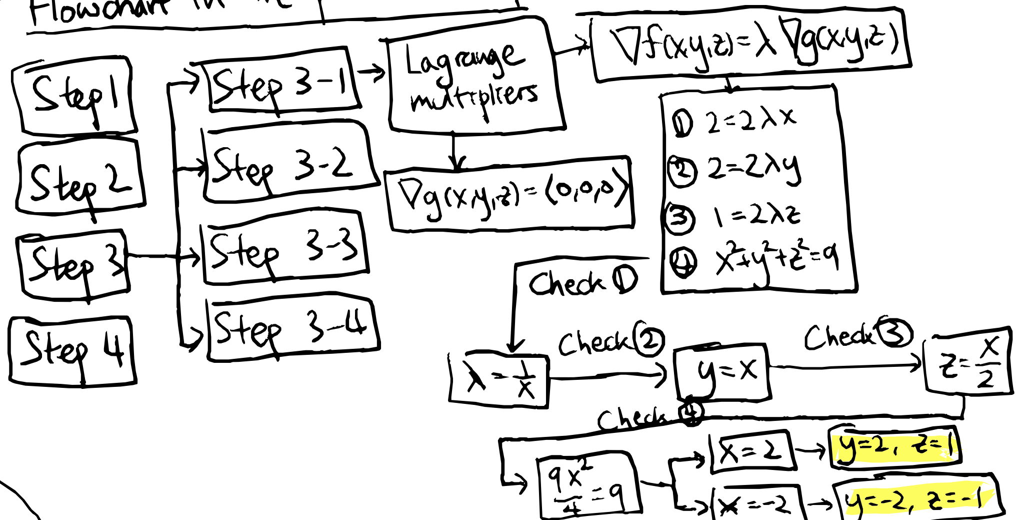
$\Rightarrow$ Maximum is 9, Minimum is -9.

Step 4 Compare.

	Critical points	Boundary
Max	X	9
Min	X	-9

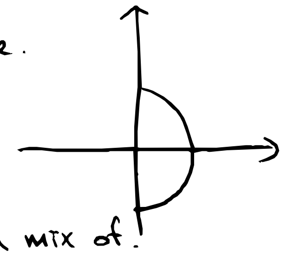
$\Rightarrow$ Maximum is 9, Minimum is -9.

Flowchart in the previous example.



<u>Example</u> Find the global max/min of $f(x,y) = x+y$ on the domain $D = \{(x,y)$ such that $x^2+y^2=1$ and $x \geq 0\}$

The domain is the right half of the unit circle.



In this example, the domain is defined using a mix of an <u>equality</u> $x^2+y^2=1$ and an <u>inequality</u> $x\geq 0$. Since there is an equality in the definition of the domain, in the 4-step process, we should look for the <u>Lagrange critical points</u>.

Let $g(x,y) = x^2+y^2$, so that the equality condition is $g(x,y)=1$.

$\boxed{\text{Step 1}}$ Find the <u>Lagrange critical points</u>.

Note that $\nabla f(x,y) = \langle 1,1\rangle$ and $\nabla g(x,y) = \langle 2x, 2y\rangle$

If $\nabla g(x,y) = \langle 0,0\rangle$, then $x=y=0$. The point $(x,y)=(0,0)$ is not on the domain $D = \{x^2+y^2=1, x\geq 0\}$, so this case is <u>out of consideration.</u>

If $\nabla f(x,y) = \lambda \nabla g(x,y)$, then $\langle 1,1\rangle = \langle 2\lambda x, 2\lambda y\rangle$, so we have 2+1 equations to solve: ① $1=2\lambda x$ ② $1=2\lambda y$ ③ $x^2+y^2=1$ (+ 1 inequality to check, ④ $x\geq 0$)

① $1=2\lambda x$ ⇝ λ and x are not zero, and $\lambda = \frac{1}{2x}$.

Plugging this into ② $1=2\lambda y$, we obtain $1 = \frac{2y}{2x}$, or $x=y$.

Plugging this into ③ $x^2+y^2=1$, we obtain $x^2+x^2=1$, or $2x^2=1$, or $x^2=\frac{1}{2}$; Since we also need to check ④ $x\geq 0$, $x=\frac{1}{\sqrt{2}}$ ⇝ $y=\frac{1}{\sqrt{2}}$.

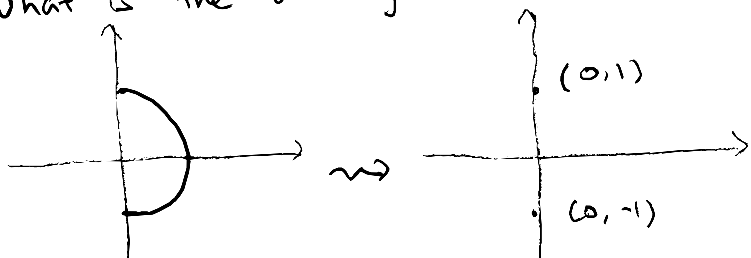
Therefore, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is the only Lagrange critical point.

$\boxed{\text{Step 2}}$ Find the max/min values of $f(x,y)$ at the Lagrange critical points.

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$ is both the max and the min.

$\boxed{\text{Step 3}}$ Find the max/min values of $f(x,y)$ on the boundary.

What is the boundary of $D = \{x^2+y^2=1,\ x \geq 0\}$?



D $\rightsquigarrow$ Boundary of D

The boundary points are $(0,1)$ and $(0,-1)$.

$f(0,1) = 0+1 = 1$ ← Max on the boundary

$f(0,-1) = 0-1 = -1$ ← Min on the boundary.

$\boxed{\text{Step 4}}$ Compare

	Lagrange critical	Boundary
Max	$\sqrt{2}$	1
Min	$\sqrt{2}$	-1

$\rightsquigarrow$ Global max: $\sqrt{2}$

Global min: -1.